The Hrushovski property and the profinite topology

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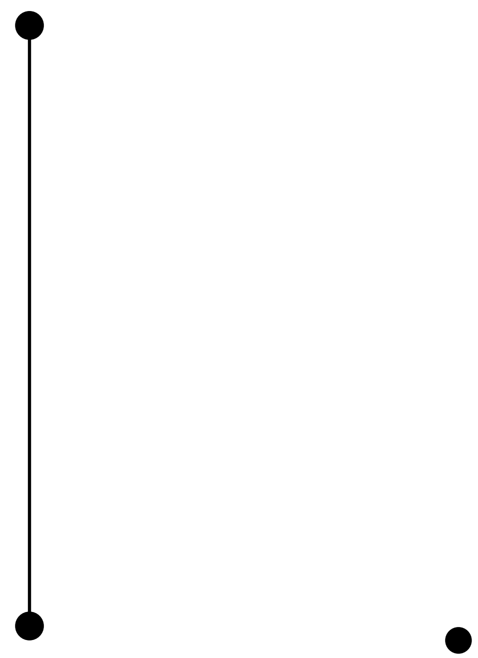
Outline

1. What is the Hrushovski property (or EPPA)?

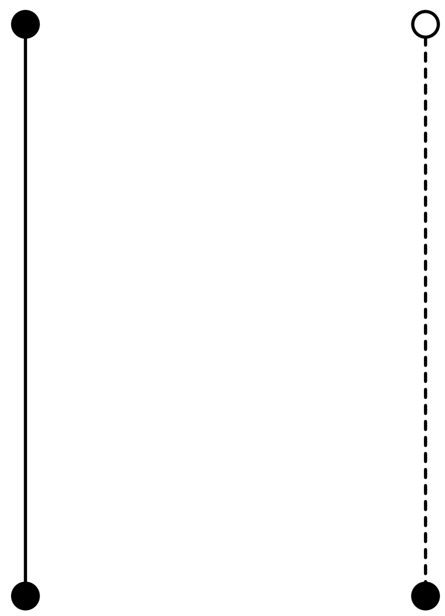
Suppose G is finite graph, and p is a ismorphism between subgraphs of G. Can p be extended to an automophism of G?

No. Consider the following.



Suppose G is finite graph, and p is a ismorphism between subgraphs of G. Is there a finite graph G' such that G embeds into G' and p be extended to an automophism of G'?

Yes.



Theorem (Hrushovski 1992, Herwig and Lascar 2000): Suppose G is a finite graph and P is the set of isomorphisms of subgraphs of G, *i.e.*, partial automorphism of G. Then there is a finite graph G' and a graph embedding $\varphi : V(G) \to V(G')$ such that for each $p \in P$, $\varphi \circ p \circ \varphi^{-1} \restriction \varphi(V(G))$ extends to an automophism of G'.

Definition: A pair (G', φ) is a **Hrushovski witness** for G.

Remark: If (G', φ) is minimal, *i.e.*, G' has the fewest vertices among witnesses, then G' is k-regular where k is the max degree among vertices in G.

Theorem (Bradley-Williams, Cameron, Evans, Hubička, Konečný, and Nešetřil (2020 and 2025)): For all $n \in \mathbb{N}$ and graphs with $|V(G)| = n$, at worst the smallest witness G' has

$$\Omega\left(2^n / \sqrt{n}\right) \leq |V(G')| \leq n2^{n-1}$$

Theorem (Hodges, Hodkinson, Lascar, and Shelah 1992): The automorphism group of the random graph has the small-index property.

Proof: Use Hrushovski's theorem among other things. □

Definition: If M is a structure and p is an isomorphism between substructures of M, then p is called a **partial automorphism.**

Definition (Hrushovski Property): Suppose $\mathcal{C}$ is a class of finite structures. $\mathcal{C}$ is said to have the **Hrushovski property**, or the **Extension Property for Partial Automorphisms** (**EPPA**), if for every $M \in \mathcal{C}$, there exists another structure $M' \in \mathcal{C}$ and an embedding $\varphi : M \to M'$ such that for all partial automorphisms p of M, we have that $\varphi \circ p \circ \varphi^{-1} \upharpoonright \varphi(M)$ extends to an automophism of M'.

Non-examples: finite linear orders, finite Boolean algebras

Examples: finite sets, finite vector spaces, finite metric spaces (Solecki 2005), finite groups (Siniora 2017)

More examples:

Definition: A **hypergraph** is a pair (V, E) where V is a set and E is a collection of subsets of V each of which contains at least two vertices.

hypergraphs

$\mathcal{L}$-structures for any finite relational language $\mathcal{L}$ (Herwig 1995)

$\mathcal{T}$-free $\mathcal{L}$-structures for any finite relational language $\mathcal{L}$ and any finite set of $\mathcal{L}$-structures $\mathcal{T}$ (Herwig and Lascar 2000)

2. Why might you be interested in EPPA?

Definition: Suppose $\mathcal{C}$ is a class of finite structures with countably many elements (up to isomorphism) and

- the hereditary property
- the joint embedding property
- the amalgamation property

Then $\mathcal{C}$ is defined to be a Fraïssé class.

Proposition: Suppose $\mathcal{C}$ is a class of finite structures with countably many elements (up to isomorphism) and

- the hereditary property
- the joint embedding property
- the Hrushovski property

Then $\mathcal{C}$ is a Fraïssé class.

Proposition (Kechris and Rosendal 2007):

Let $\mathcal{K}$ be a Fraïssé class, K its Fraïssé limit, and $\mathrm{Aut}(K)$ the automorphism group of K. Then $\mathcal{K}$ has the Hrushovski property if and only if there is a countable chain $C_0 \leq C_1 \leq C_2 \leq \ldots \leq \mathrm{Aut}(K)$ of compact subgroups whose union is dense in $\mathrm{Aut}(K)$.

Proposition: Suppose $\mathcal{K}$ is a Fraïssé class and K its Fraïssé limit. If $\mathcal{K}$ has both the Hrushovski property and the amalgamation property with automophisms, then $\mathrm{Aut}(K)$ has ample generics.

3. How is EPPA connected to the topology of free groups?

3.1 Topology of free groups

Definition: If F is a finitely generated free group,

$$\{wH : w \in F, H \leq F, [F : H] < \infty\}$$

defines a basis for the **profinite topology** on F.

Theorem: (Hall 1949) Finitely generated subgroups are closed.

Theorem: (Ribes and Zalesskii 1992) If $H_1, ..., H_n$ are finitely generated subgroups of F, then

$$H_1 ... H_n = \{h_1 ... h_n : h_i \in H_i\}$$

is closed in the profinite topology on F.

Remark: There are many proofs of R and Z, including from EPPA

3.1 Topology of free groups

Definition: Given non-empty set of primes L, a subgroup H of G is an ***L*-index** subgroup of G if $[G : H]$ can be written as a finite product of primes in L.

Definition: Given a finitely generated free group F and a non-empty set of primes L, the **pro-*L* topology** on F is the topology with basis given by cosets of L-index normal subgroups of F.

Definition: If L is the set of all primes, the pro-L topology is called the **profinite topology**.

Definition: If L is the set of all odd primes, the pro-L topology is called the **pro-odd topology**.

3.1 Topology of free groups

Definition: A **pseudovariety $\mathcal{C}$ of finite groups** is a non-empty class of finite groups with the following three properties:

1. if $G \in \mathcal{C}$ and $H \leq G$, then $H \in \mathcal{C}$;
2. if $G \in \mathcal{C}$ and $G \longrightarrow H$ is a surjective homomorphism of groups, then $H \in \mathcal{C}$; and
3. if $G, H \in \mathcal{C}$, then $G \times H \in \mathcal{C}$.

Definition: For any group G and any pseudovariety $\mathcal{C}$, the **pro-$\mathcal{C}$ topology on G** is the coasest topology on G with which G a is topological group and any normal subgroup N of G with $G/N \in \mathcal{C}$ is open.

Example: Given a non-empty set of primes L, the class of finite groups with order given by a finite product of primes in L is a pseudovariety $\mathcal{C}$, and the pro-$\mathcal{C}$ topology = the pro-L topology

3.1 Topology of free groups

Theorem: (Ribes and Zalesskii 1993) Let $\mathcal{C}$ be a pseudovariety of groups and F be a finitely generated free group. If $H_1, \ldots, H_n$ are finitely generated closed subgroups of F, then

$$H_1 \ldots H_n = \{h_1 \ldots h_n : h_i \in H_i\}$$

is closed in the pro-$\mathcal{C}$ topology on F.

Remark: Only one proof is known.

3.2 Herwig and Lascar

Definition: A **tournament** is a complete graph with an orientation on all the edges.

Definition: Given a group G and a subgroup H, H is **closed under square roots** if for all $g \in G$, $g^2 \in H$ implies $g \in H$.

Remark: If H is closed in the pro-odd topology, then H is closed under square roots.

3.2 Herwig and Lascar

Theorem (Herwig and Lascar 2000): TFAE:

- the class of finite tournaments has the Hrushovski property;
- for any finitely generated free group F and any finitely generated subgroup $H \leq F$, if H is closed under square roots, then H is closed in the pro-odd topology.

Proof: Uses Ribes and Zalesskii. □

Theorem (Siniora 2017): The universal homogeneous tournament has ample generics if and only if the class of finite tournaments has Hrushovski property.

3.2 Herwig and Lascar

Definition: Suppose $L \subset \mathbb{N}$. An ***L*-hypertournament** is a structure with universe V and exactly one l-ary relation for each $l \in L$. Moreover, each relation satisfies the following properties.

1. For each $l \in L$ and $\{v_1, ..., v_l\} \in [V]^l$, there exists $\sigma \in \mathrm{Sym}_l$ with $\left(v_{\sigma(1)}, ..., v_{\sigma(l)}\right) \in R_l$.
2. There is no $l \in L$ and $\{v_1, ..., v_l\} \in [V]^l$ such that $(v_1, v_2, ..., v_l), (v_2, v_3, ..., v_1), ..., (v_l, v_1, ..., v_{l-1}) \in R_l$.

3.2 Herwig and Lascar

Definition: Given a group G and a subgroup H, H is **closed under l roots** if for all $g \in G$, $g^l \in H$ implies $g \in H$.

Definition: Given a group G, a subgroup H, and $L \subset \mathbb{N}$, H is **closed under L roots** if for all $l \in L$, H is closed under l roots.

Notation: If L is a subset of the primes, $L^\perp$ is the set of other primes.

Remark: If H is closed in the pro-$L^\perp$ topology, then H is closed under L roots.

3.2 Herwig and Lascar

Theorem (Huang, Pawliuk, Sabok, and Wise 2018): Given a non-empty proper subset of primes L, TFAE:

- the class of finite L-hypertournaments has the Hrushovski property;
- for any finitely generated free group F and any finitely generated subgroup $H \leq F$, H is closed under L roots implies H is closed in the pro-$L^{\perp}$ topology.

Proof: Uses Ribes and Zalesskii. $\square$

Theorem (Huang, Pawliuk, Sabok, Wise 2018): Suppose L is a set of all primes but one. The class of L-hypertournaments does not have the Hrushovski property.

Theorem: (Hrushovski 1992, Herwig and Lascar 2000) Suppose G is a finite graph and P is the set of isomorphisms of subgraphs of G, *i.e.*, partial automorphism of G. Then there is a finite graph G' and a graph embedding $\varphi : G \to G'$ such that for each $p \in P$, $\varphi \circ p \circ \varphi^{-1} \restriction \varphi(G)$ extends to an automophism of G'.

The following proof comes from Herwig and Lascar.

We need to construct G' and φ, and define extensions of each $p \in P$ on G'. Consider the free group $F(P)$. Let $H < F(P)$.

$$V(G') := F(P)/H$$

Fix a vertex v_0 of G. For all $v \in V(G)$, let $\varphi(v) := pH$ where $p \in P$ and $p(v_0) = v$.

$$E(G') := \{\text{edges from } G\} \cup \{\text{new edges}\}$$
$$\text{edges from } G = \{\{\varphi(v), \varphi(v')\} : \{v, v'\} \in E(G)\}$$
$$\text{new edges} = \{w \cdot e : w \in F(P), e \text{ is an edge from } G\}$$

For all $wH \in V(G')$ and $p \in P$, $\varphi \circ p \circ \varphi^{-1}(wH) := pwH$.

Requirements for H:

- H must make φ a well-defined function.
- H must make φ is injective.
- H can't break the fact that φ is a graph embedding.

3.3 Proofs

First attempt at H. Consider

$$H_{v_0} := \{p_n \cdot \dots \cdot p_1 \in F(P) : p_n \circ \dots \circ p_1(v_0) = v_0\}.$$

Note H_{v_0} is the first fundamental group of the graph with $V(\Gamma) := V(G)$ and $E(\Gamma) := \{(v_1, v_2) : \exists p \in P, p(v_1) = v_2\}$

Pros of H_{v_0}

- H_{v_0} makes φ a well-defined function

Cons of H_{v_0}

- φ is maybe not injective
- $\left[F(P) : H_{v_0}\right]$ might be infinite

Second attempt at H. Define $B := \{(p, q) \in P^2 : p(v_0) \neq q(v_0)\}$. For $(p, q) \in B$, $p^{-1} \cdot q \notin H_{v_0}$. So, we can define a normal subgroup of finite index $N_{p,q}$ with $p^{-1} \cdot qN_{p,q} \cap H_{v_0} = \emptyset$.

Let $N_B := \bigcap_{(p,q)\in B} N_{p,q}$.

Consider $H_2 := H_{v_0} N_B$.

Pros of H_2

- H_2 makes φ a well-defined function
- H_2 makes φ a injective
- $[F(P) : H_2]$ is finite

Cons of H_2

- the new edges may still break the embedding property of φ

Third attempt at H. We need to make sure that non-edges in $\varphi(G)$ are not in any orbit of any edge in $\varphi(G)$. Let B' be the set of all $(p_1, p_2, p_3, p_4) \in P^4$ such that

$$p_1 p_3^{-1} p_4 p_2^{-1} \notin p_1 H_{v_0} p_1^{-1} p_2 H_{v_0} p_2^{-1}$$

and

$$p_1 p_4^{-1} p_3 p_2^{-1} \notin p_1 H_{v_0} p_1^{-1} p_2 H_{v_0} p_2^{-1}.$$

Apply Ribes and Zalesskii.

Let $H = H_{v_0}\left(\bigcap_{(p,q)\in B} N_{p,q} \cap \bigcap_{(p_1,p_2,p_3,p_4)\in B'} N_{p_1,p_2,p_3,p_4}\right)$

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